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# Cauchy-Schwarz-Bunyakovsky Inequality

In this journal artical we discuss the proof on page 186 of Linear Algebra by, Otto Bretscher. We show a detailed proof of the Cauchy-Schwarz-Bunyakovsky inequality, and give a brief background of the people involved in the proof.

Augustin-Louis Cauchy was born in Paris, France, on August 21, 1789; it was his work that others followed to discover this theorem. However, the theorem was not proven in his lifetime. He died in Paris on May 22, 1857.

Hermann Amandus Schwarz, born in Germany, lived from January 25, 1843, to November 30, 1921. Viktor Yakovlevich Bunyakovsky, a Russian, lived from December 16,1804 to December 12, 1889. The interesting aspect of these two mathematicians and this theorem is, both followed the same work of Cauchy, and both independently discovered the same proof, at roughly the same time. Schwartz was publicized first and therefore got a little more praise and credit; and so most books will refer to this theorem as the Cauchy-Schwarz inequality.

Cauchy-Schwarz-Bunyakovsky Inequality

We would like to prove that, if **x** and **y** are arbitrary vectors in $R^N$, then

$|\mathbf{x}| \cdot |\mathbf{y}| \geq |\mathbf{x} \cdot \mathbf{y}|$. where $|\mathbf{x}|$ is the absolute value of **x.**

We will now show a detailed proof of this theorem, and we use notation found in, Fundamental Structures of Algebra, by Mostow, Sampson, and Meyer.

We use proven facts on page 199 of Mostow, Sampson, and Meyer.

notation: $(\mathbf{x},\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

**11.2** $|t\mathbf{x}|^2 = t^2|\mathbf{x}|$ (t any real number)
**11.3** $|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + 2(\mathbf{x},\mathbf{y}) + |\mathbf{y}|^2$
**11.5** $(t\mathbf{x} , \mathbf{y}) = t \cdot(\mathbf{x} , \mathbf{y})$ and $(\mathbf{x} + \mathbf{y}, \mathbf{z}) = (\mathbf{x} , \mathbf{z}) + (\mathbf{y} , \mathbf{z})$
for any vectors **x,y,z** and any real number t.

consider:
$|\mathbf{x}|^2 \cdot |\mathbf{y}|^2 \geq (\mathbf{x} \cdot \mathbf{y})^2$

*proof.* Replace **x** in (11.3) by t , t being any real number. Using (11.2) and (11.5), we have t**x** **y**| = $t^2$ **x**| + 2t(**x,y** |**y** $^2$. We temporarily write a = **x**| ,b = 2(**x** **y**), and c |**y** $^2$, because we wish to sow the quadratic relationship,

|t + **y** $^2$ 0, because the square magnitude must be positive, we have then

at + bt + c ≥

and this must hold for any real number t, because of the cosistency from above. Consequently this quadratic polynomial cannot have two distinct real roots because a = **x**| and is therefore positive (sq), and c =| $|^2$ determinant ac ≥ $^2$, or 0 $b^2$

Now let **x** **y** be any two nonzero vectors in our vector space V.

≥ 4 ( $)^2$ |**x** $^2$ | $|^2$ ⇒ which is 0 $b^2$

or

| $|^2$ **y**| ≥ **x, y**)

we take the square root;

| | | | | **x**, )| |( , **y** | = absolute value of ( ,**y**

This is what we wished to show; we have proven the Cauchy-Schwartz-Bunyakovsky Inequality theorem.

# References

1) Fundamental Structures of Algebra, by Mostow, Sampson, and Meyer. McGraw-Hill Book Company, inc.; New York Toronto London.

2) Linear Algebra, by Sterling K. Berberian. Oxford University Press; Oxford New York Tokyo. 1992

3) Linear Algebra with Applications, by Otto Bretscher; Prentice Hall, Upper Saddle tiver, New Jersey 07458; 1997.